LINDELÖF TYPE OF GENERALIZATION OF SEPARABILITY IN BANACH SPACES

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ABSTRACT. We will introduce the countable separation property (CSP) of Banach spaces X, which is defined as follows: For each set $\mathcal{F} \subset \mathrm{X}^*$ such that $\overline{\mathrm{span}}^{\omega^*}(\mathcal{F}) = \mathrm{X}^*$ there is a countable subset $\mathcal{F}_0 \subset \mathcal{F}$ with $\overline{\mathrm{span}}^{\omega^*}(\mathcal{F}_0) = \mathrm{X}^*$. All separable Banach spaces have CSP and plenty of examples of non-separable CSP spaces are provided. Connections of CSP with Markučevič-bases, Corson property and related geometric issues are discussed.

1. Introduction

In this article we study a class of relatively small non-separable Banach spaces. If X is a Banach space and X^* is its dual, then a subset $\mathcal{F} \subset \mathrm{X}^*$ is said to *separate* X if for all $x \in \mathrm{X} \setminus \{0\}$ there is $f \in \mathcal{F}$ such that $f(x) \neq 0$. We investigate the Banach spaces X, which have the following *countable separation property* (CSP):

$$\text{Whenever } \mathcal{F} \subset \mathrm{X}^* \text{ separates X, there exists a countable separating subset } \mathcal{F}_0 \subset \mathcal{F}.$$

It turns out that all separable Banach space have CSP but there exist also numerous examples of non-separable CSP spaces (see the last section for summary). The definition of the countable separation property resembles that of the Lindelöf property. In fact it turns out that if the unit sphere $\mathbf{S}_{\mathrm{X}}$ of a Banach space X is weakly Lindelöf, then X has CSP. We will point out various connections between CSP and the established theory of non-separable spaces. For example, if X is weakly compactly generated and has CSP, then it follows that X is already separable.

This study is also closely related to [4], in which certain generalizations of separability, the so called *Kunen-Shelah properties*, are summarized and discussed. The chain of implications

$$\text{separability} \implies \mathrm{KS}_7 \implies \mathrm{KS}_6 \implies \ldots \implies \mathrm{KS}_0$$

was proved there. The only known examples of non-separable spaces having any of the properties $\mathrm{KS}_7 - \mathrm{KS}_2$ are constructed under set theoretic axioms extraneous to ZFC. To mention in this context the most important properties from the list, KS_4, KS_2 or KS_1 of X means that there is *no* closed subspace $\mathrm{Y} \subset \mathrm{X}$ admitting an ω_1-polyhedron, an uncountable biorthogonal system or an uncountable M-basis, respectively, in Y. It turns out that $\mathrm{KS}_4 \implies \mathrm{CSP} \implies \mathrm{KS}_1$.

In order to motivate the introduction of CSP concept, let us discuss the Kunen-Shelah properties a little further. One could ask whether some of the Kunen-Shelah properties of Banach spaces are in fact equivalent to separability. Recently Todorčević [19] answered some old questions about the existence of bases in non-separable Banach spaces. In particular, he proved that it is consistent that each

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non-separable Banach space admits an uncountable biorthogonal system and hence it is consistent that $\mathrm{KS}_2 \implies \mathrm{CSP}$. Actually, by combining the results in [13, p.1086-1099] and [19] one can see that the question about the equivalence of separability with KS_2 is independent of ZFC. This positions the class of CSP spaces interestingly with respect to the Kunen-Shelah properties. On one hand, CSP is a class of Banach spaces close to KS_2 and still containing absolute non-separable representatives. On the other hand, it turns out that many of the known interesting examples of non-separable Banach spaces (both in and outside ZFC) fall into CSP class.

General concepts and notations. Real Banach spaces are typically denoted by $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$. We denote by $\mathbf{B}_{\mathrm{X}}$ the closed unit ball of X and by $\mathbf{S}_{\mathrm{X}}$ the unit sphere of X. Unless otherwise stated, we will apply cardinal arithmetic notations.

See [3], [6] and [13] for the standard notions in set-theory, Banach spaces and topology, respectively. We refer to Zizler's survey [21] on the non-separable Banach spaces for most of the definitions and results used here.

Recall that $\mathcal{F} \subset \mathrm{X}^*$ is a separating subset if and only if for each $x \in \mathrm{X}$ there is $f \in \mathcal{F}$ such that $f(x) \neq 0$ if and only if $\overline{\mathrm{span}}^{\omega^*}(\mathcal{F}) = \mathrm{X}^*$ (see e.g. [6, p.55]). Let X be a Banach space and let $\mathcal{F} = \{(x_\alpha, x^*_\alpha)\}_\alpha \subset \mathrm{X} \times \mathrm{X}^*$ be a biorthogonal system, i.e. $x^*_\alpha(x_\beta) = \delta_{\alpha,\beta}$. If $\overline{\mathrm{span}}(x_\alpha) = \mathrm{X}$ and $\overline{\mathrm{span}}^{\omega^*}(x^*_\alpha) = \mathrm{X}^*$, then $\mathcal{F}$ is called a *Markučevič-basis* or *M-basis*. Recall that each separable Banach space admits an M-basis, see [6, p.219]. An M-basis $\{(x_\alpha, f_\alpha)\}_\alpha \subset \mathrm{X} \times \mathrm{X}^*$ is called countably λ-norming if there is $\lambda \geq 1$ such that for each $x \in \mathrm{X}$ there is $f \in \mathrm{X}^*$ satisfying $\lambda ||x|| \leq f(x)$ and $|\{\alpha :\ f(x_\alpha) \neq 0\}| \leq \omega$. If X admits a countably norming M-basis, then X is called *Plichko* (reformulation according to [20]). It is said that X has the *density property* (DENS) if $\omega^*-\mathrm{dens}(\mathrm{X}^*) = \mathrm{dens}(\mathrm{X})$.

A compact Hausdorff space K is called a *Corson compact* if it can be embedded in a Σ-product of real lines, and a Banach space X is Weakly Lindelöf Determined (WLD) if $(\mathbf{B}_{\mathrm{X}^*}, \omega^*)$ is a Corson compact. To mention a more general notion, recall that a compact Hausdorff space K is a *Valdivia compact* if there is an embedding $h\colon K \to \mathbb{R}^\Gamma$ (where $\mathbb{R}^\Gamma$ is equipped with the product topology) such that $h(K) \cap A$ is dense in $h(K)$, where $A \subset \mathbb{R}^\Gamma$ is the corresponding Σ-product. We will call a topological space *countably perfect* if each point is in the countable closure of the rest of the space.

The following folklore facts will be applied frequently: Suppose that (T, τ) is a countably tight topological space, λ is an ordinal with $\mathrm{cf}(\lambda) > \omega$ and $\{E_\alpha\}_{\alpha<\lambda}$ is a family of closed subsets of T such that $E_\alpha \subset E_\beta$ for $\alpha < \beta$. Then $\overline{\bigcup_{\alpha<\lambda} E_\alpha}^{\tau} = \bigcup_{\alpha<\lambda} E_\alpha$. A subset $\mathcal{F} \subset \mathrm{X}^*$ separates X if and only if $\overline{\mathrm{span}}^{\omega^*}(\mathcal{F}) = \mathrm{X}^*$ (see e.g. [6, p.55]).

Fact 1.1. *Let* $q\colon \mathrm{X} \to \mathrm{X}/\mathrm{Y}$ *be the quotient map* $q\colon x \mapsto x + \mathrm{Y}$. *Then* $\overline{q(A)} = \{z + \mathrm{Y} | z \in \overline{A + \mathrm{Y}}\}$ *for any subset* $A \subset \mathrm{X}$.

Proof. The condition that $x \in \overline{A + \mathrm{Y}}$ is equivalent to

$$\inf_{y \in \mathrm{Y},\ a \in A} ||x - (a + y)|| = \mathrm{dist}_{\mathrm{X}/\mathrm{Y}}(q(x), q(A)) = 0.$$

□

2. General properties

It is easy to see that the following formulations of CSP are equivalent:

(2.1)

- X has CSP: if $\mathcal{F} \subset \mathrm{X}^*$ separates X, then there is a countable $\mathcal{F}_0 \subset \mathcal{F}$, which separates X.
- For each ω^*-dense linear subspace $V \subset \mathrm{X}^*$ there exists a countable subset $\mathcal{F}_0 \subset V$ such that $\overline{\mathrm{span}}^{\omega^*}(\mathcal{F}_0) = \mathrm{X}^*$.
- Each family of closed subspaces with trivial intersection has a countable subfamily with trivial intersection.
- There does not exist an uncountable separating family $\mathcal{F} \subset \mathrm{X}^*$ such that each separating subfamily $\mathcal{F}_0 \subset \mathcal{F}$ has the same cardinality as $\mathcal{F}$.

Note that in particular $\omega^*-\mathrm{dens}(\mathrm{X}^*) = \omega$ if X has CSP.

The spaces $c_0(\Gamma),\ \ell^p(\Gamma),\ 1 \leq p \leq \infty,\ |\Gamma| \geq \omega_1$, provide examples of spaces without CSP, since $\{e_\gamma^*\}_{\gamma\in\Gamma}$ is an uncountable minimal separating family.

The following fact appeared already in [1] with a different formulation.

Proposition 2.1. *Separable Banach spaces have CSP.*

This follows also immediately from the following observation.

Proposition 2.2. *Let* X *be a Banach space such that* $\mathbf{S}_\mathrm{X}$ *is Lindelöf in the relative weak topology. Then X has CSP.*

Proof. Let $\{f_\gamma\}_{\gamma\in\Gamma} \subset \mathrm{X}^*$ be a separating family of functionals. Then $\bigcup_{\gamma\in\Gamma} f_\gamma^{-1}(\mathbb{R}\setminus\{0\}) = \mathrm{X}\setminus\{0\}$. In particular, the subsets $U_\gamma = f_\gamma^{-1}(\mathbb{R}\setminus\{0\}) \cap \mathbf{S}_\mathrm{X},\ \gamma \in \Gamma$, define a ω-open cover for $\mathbf{S}_\mathrm{X}$. According to the Lindelöf property of $(\mathbf{S}_\mathrm{X}, \omega)$ there exists a countable subcover $\{U_{\gamma_n}\}_{n<\omega} \subset \{U_\gamma\}_{\gamma\in\Gamma}$. We conclude that $\{f_{\gamma_n}\}_{n<\omega}$ is a separating family of X. □

We will subsequently give some examples of non-separable CSP spaces. It is straightforward to verify that CSP is preserved under isomorphisms.

Proposition 2.3. *Let* X *be a Banach space with CSP and let* $\mathrm{Y} \subset \mathrm{X}$ *be a closed subspace. Then* Y *has CSP.*

Proof. Assume to the contrary that Y fails CSP. Suppose that $\mathcal{F} \subset \mathrm{Y}^*$ is an uncountable minimal separating set for Y. Let $\widetilde{\mathcal{F}} \subset \mathrm{X}^*$ be a set of functionals obtained from $\mathcal{F}$ by Hahn-Banach extension. Then $\mathrm{Y}^\perp \cup \widetilde{\mathcal{F}}$ is an uncountable minimal separating subfamily for X; a contradiction. □

Example 2.4. *The space* ℓ^∞ *does not have CSP.*

The space ℓ^∞ contains an isometric copy of $\ell^1(2^\omega)$ (see [6, p.86]). Clearly $\ell^1(2^\omega)$ does not have CSP. Thus the claim follows by applying Proposition 2.3 that CSP is inherited by the closed subspaces.

The preceding example shows that there is a countable $\mathcal{F}_0 \subset (\ell^\infty)^*$ such that

$$\mathrm{span}(\{f|_{c_0} :\ f \in \mathcal{F}_0\}) \text{ is dense in } (\ell^1, \omega^*),$$

and

$$\mathrm{span}(\mathcal{F}_0) \text{ is not dense in } ((\ell^\infty)^*, \omega^*),$$

even though ℓ^1 is ω^*-dense in $(\ell^\infty)^*$ by Goldstine's theorem.

Example 2.5. The spaces JL_0, JL_2 (see [9]) have CSP according to subsequent observations (see Proposition 3.1), but $JL_0/c_0 = c_0(2^\omega)$ and $JL_2/c_0 = l^2(2^\omega)$ (see e.g. [21, p. 1757]) clearly do not. We conclude that

(i) CSP does not pass to quotients in general.
(ii) The dual of a CSP space may contain a ω^*-closed subspace, which is not ω^*-separable.

The spaces JL_0 and JL_2 are not Lindelöf in their weak topology (see e.g. [21, p.1757,1764]).

If X and Y have CSP, does it follow that $\mathrm{X} \oplus \mathrm{Y}$ has CSP? If so, is CSP a three-space property, i.e. does X have CSP whenever X/Y and $\mathrm{Y} \subset \mathrm{X}$ have CSP? We will give a partial answer to this problem in Theorem 4.3.

Even though CSP is not a sufficient condition for separability, it is still quite a strong condition 'towards separability'.

Proposition 2.6. *Let* X *be a Banach space such that* X^* *has CSP. Then* X *is separable.*

Proof. Clearly $\mathrm{X} \subset \mathrm{X}^{**}$ embedded canonically separates X^*. Then according to CSP one can find a sequence $(x_n)_{n\in N} \subset \mathrm{X}$ separating X^*. We claim that $\overline{\mathrm{span}}(x_n) = \mathrm{X}$. Indeed, if this was not the case, then one could find by the Hahn-Banach theorem a non-zero functional $f \in \mathrm{X}^*$ vanishing on $\overline{\mathrm{span}}(x_n)$. But this is not possible, since $(x_n) \subset \mathrm{X}^{**}$ separates X^*. Thus X is separable. $\square$

As a brief remark we would like to mention a mild condition under which separability and CSP are equivalent.

Let us consider the case that X admits a system $(\{x_\alpha\}_\alpha, \{f_\beta\}_\beta) \in \mathcal{P}(\mathrm{X}) \times \mathcal{P}(\mathrm{X}^*)$ satisfying the following conditions:

$$\begin{cases} \overline{\mathrm{span}}(\{x_\alpha\}_\alpha) = \mathrm{X} \\ \overline{\mathrm{span}}^{\omega^*}(\{f_\beta\}_\beta) = \mathrm{X}^* \\ \text{For each } \beta \text{ we have } |\{\alpha | f_\beta(x_\alpha) \neq 0\}| \leq \omega^*\mathrm{-dens}(\mathrm{X}^*). \end{cases} \tag{2.2}$$

For a Banach space X the DENS property or the existence of an M-basis clearly provide a system satisfying (2.2).

Proposition 2.7. *A Banach space* X *is separable if and only if it has CSP and admits a system satisfying* (2.2). *In particular each CSP space with an M-basis is separable.*

Proof. Suppose that X is a CSP space, and the system $(\{x_\alpha\}_\alpha, \{f_\beta\}_\beta)$ satisfies (2.2). Then there exists a countable separating subfamily $\mathcal{F} \subset \{f_\beta\}_\beta$, since X has CSP. Thus

$$|\{\alpha | x_\alpha \neq 0\}| = |\{\alpha | f(x_\alpha) \neq 0 \text{ for some } f \in \mathcal{F}\}| \leq |\mathcal{F} \times (\omega^*\mathrm{-dens}(\mathrm{X}^*))| = \omega.$$

Hence X is separable. On the other hand, each separable Banach space X has an M-basis (see e.g. [6, p.219]) and hence system (2.2). Recall that, by Proposition 2.1, separability of X implies CSP. $\square$

Example 2.8. *For a Valdivia compact K the space $C(K)$ has CSP if and only if $C(K)$ is separable. If a Banach space* X *is Plichko and has CSP, then* X *is separable.*

Indeed, if K is Valdivia, then $C(K)$ is Plichko (see [7]). Any Plichko space admits an M-basis, so that Proposition 2.7 yields that X is separable.

Proposition 2.9. *Let* X *be a CSP space and* $C \subset \mathrm{X}$ *a weakly compact set. Then* C *is separable.*

Proof. Observe that $\mathrm{Y} = \overline{\mathrm{span}}(C)$ is a WCG subspace and hence admits an M-basis. According to Proposition 2.3 the space Y has CSP. Thus, by Proposition 2.7 Y is separable and so is C. □

3. Topological point of view

A topological space T is called *dense-separable* if each dense subset $A \subset T$ is separable (see [11] for discussion).

Recall the following concept due to Corson and Pol. A Banach space X has property (C) if, for each family $\mathcal{A}$ of closed convex subsets of X having empty intersection, there exists a countable subfamily $\mathcal{A}_0 \subset \mathcal{A}$ with empty intersection. Pol gave an important characterization of property (C) in terms of a dual space formulation (C'), which is a kind of convex version of countable tightness (see [14, Thm.3.4]). This condition appears in the result below:

Proposition 3.1. *Let* X *be a Banach space. Consider the following conditions:*

(1) (X^*, ω^*) *is countably tight i.e. for each* $a \in \overline{A}^{\omega^*} \subset \mathrm{X}^*$ *there is a subset* $(a_n)_{n<\omega} \subset A$ *such that* $a \in \overline{\{a_n | n < \omega\}}^{\omega^*}$.
(2) X^* *is dense-separable in the* ω^**-topology.*
(3) X *satisfies property* (C')*: for each* $A \subset \mathrm{X}^*$ *and* $f \in \overline{A}^{\omega^*}$ *there is a countable* $A_0 \subset A$ *such that* $f \in \overline{\mathrm{conv}}^{\omega^*}(A_0)$.
(4) *For each* $A \subset \mathrm{X}^*$ *and* $f \in \overline{A}^{\omega^*}$ *there is countable* $A_0 \subset A$ *such that* $f \in \overline{\mathrm{span}}^{\omega^*}(A_0)$.
(5) X *has CSP i.e. for each* $A \subset \mathrm{X}^*$ *such that* $\overline{\mathrm{span}}^{\omega^*}(A) = \mathrm{X}^*$ *there is a countable* $A_0 \subset A$ *such that* $\overline{\mathrm{span}}^{\omega^*}(A_0) = \mathrm{X}^*$.

If X^* *is* ω^**-separable, then* $1 \implies 2 \implies 5$ *and* $1 \implies 3 \implies 4 \implies 5$.

Proof. Let us first check that implication $(1) \implies (2)$ holds if X^* is ω^*-separable. Let X^* be ω^*-separable space satisfying (1) and let $A \subset \mathrm{X}^*$ be a ω^*-dense subset. Fix a ω^*-dense subset $(x_k)_{k<\omega} \subset \mathrm{X}^*$. By using the countable tightness of (X^*, ω^*), we may pick $\{a_k^{(n)} | n, k < \omega\} \subset A$ such that $x_k \in \overline{\{a_k^{(n)} | n < \omega\}}^{\omega^*}$ for each $k < \omega$. Hence $\overline{\{a_k^{(n)} | n, k < \omega\}}^{\omega^*} = \mathrm{X}^*$, so that (2) holds, as A was arbitrary.

Let us check the implication $(4) \implies (5)$ for ω^*-separable X^*. First recall (2.1). Let X^* be ω^*-separable space satisfying (4) and let $(x_n)_{n<\omega} \subset \mathrm{X}^*$ be a ω^*-dense subset in X^*. Consider a separating family $A \subset \mathrm{X}^*$. Thus $\mathrm{span}(A)$ is ω^*-dense in X^*. According to condition (4) we can find a countable set $C_n \subset \mathrm{span}(A)$ for each $n < \omega$ such that $x_n \in \overline{\mathrm{span}}^{\omega^*}(C_n)$. Thus $\overline{\mathrm{span}}^{\omega^*}(\bigcup_n C_n) = \mathrm{X}^*$. Note that each of the sets C_n is contained in the linear span of countably many vectors of A. Therefore there is countable $A_0 \subset A$ such that

$$\overline{\mathrm{span}}^{\omega^*}(A_0) = \overline{\mathrm{span}}^{\omega^*}(\bigcup_n C_n) = \mathrm{X}^*,$$

so that X satisfies (5). Other implications are clear. □

We do not know if (2) $\implies$ (3) above or if some implications can be reversed.

For example the following spaces are non-separable and have CSP according to Proposition 3.1: JL_0, JL_2 and $C(K)$, where K is the double-arrow space. Indeed, these spaces have property (C) and a ω^*-separable dual (see e.g. [21, p.1757], [14, p.146]).

Assuming CH Kunen constructed an interesting Hausdorff compact K, which is separable, scattered and non-metrizable (see [13, p.1086-1099] for discussion). Kunen's $C(K)$ space is non-separable and $(C(K)^*, \omega^*)$ is hereditarily separable (see [5, p.476]), hence separable and countably tight.

Example 3.2. *There is a bounded injective linear operator* $T\colon JL_0 \to c_0(2^\omega)$ *whose range is non-separable.*

In justifying this we will apply the fact that JL_0 contains c_0 and that $JL_0/c_0 = c_0(2^\omega)$ (see [21, p. 1757]). Let $(e_n)_{n<\omega} \subset c_0$ be the canonical unit vector basis and let $(e_n^*)_{n<\omega} \subset \ell^1$ be the corresponding functionals. Let $(f_n)_{n<\omega} \subset JL_0^*$ be the Hahn-Banach extension of $(e_n)_{n<\omega}$. Define $S\colon JL_0 \to c_0$ by $x \mapsto ((n+1)^{-1} f_n(x))_{n<\omega}$. Let $q\colon JL_0 \to JL_0/c_0$ be the canonical quotient mapping. Then $T\colon x \mapsto (S(x), q(x))$ defines the required map $JL_0 \to c_0(\omega) \oplus c_0(2^\omega) = c_0(2^\omega)$.

Problem 3.3. *Suppose that* X *admits a long unconditional basis and* $\mathrm{Y} \subset \mathrm{X}$ *is a closed subspace having CSP. Does it follow that* Y *is separable?*

Dual CSP spaces can be characterized as follows:

Theorem 3.4. *Let* X *be a Banach space. Then the following are equivalent:*

(1) X^* *has CSP*
(2) X *is separable and* X^* *has property (C)*
(3) X *is separable and does not contain* ℓ^1 *isomorphically.*

Proof. The equivalence of the last two conditions is known (see [21, Thm.4.2]).

If (2) holds, then X^{**} is ω^*-separable by Goldstine's theorem, so that Proposition 3.1 can be applied together with property (C) to obtain that X^* has CSP.

On the other hand, if X^* has CSP, then by Proposition 2.6 we known that X must be separable. Now assume to the contrary that X contains an isomorphic copy of ℓ^1. Then it is known that X^* contains a complemented isomorphic copy of ℓ^∞. Since ℓ^∞ does not have CSP it follows by Proposition 2.3 that X^* fails CSP, a contradiction. Thus X does not contain ℓ^1. $\square$

For example the James Tree and the James function space (see [12]) are separable spaces not containing ℓ^1, and whose duals JT^* and JF^* are non-separable CSP spaces.

3.1. $C(K)$ spaces.

Proposition 3.5. *Let* L *be a locally compact Hausdorff space such that* $C_0(L)$ *has CSP. Then* L *is dense-separable and the interior of the derived set of* L *is countably perfect.*

Proof. Recall that locally compact Hausdorff spaces are completely regular. If the interior of the derived set of L is non-empty, then let $x \in \mathrm{int}(D(L))$. If such x exists, it is not an isolated point in L, and in any case we may fix a dense subset $\Gamma \subset L$ such that $x \notin \Gamma$. Consider the point evaluation maps $f \overset{\delta_k}{\mapsto} f(k)$ in $C_0(L)^*$ for $k \in \Gamma$.

Clearly these maps separate $C_0(L)$. Since $C_0(L)$ has CSP, there exists a countable subset $\Gamma_0 \subset \Gamma$ such that the associated evaluation maps still separate $C_0(L)$. We claim that $L = \overline{\Gamma}_0$. Indeed, if there exists a point $y \in L \setminus \overline{\Gamma}_0$, then by the completely regularity of L there exists $f \in C_0(L)$ attaining value 1 at y but vanishing on $\overline{\Gamma}_0$. This contradicts the fact that the point evaluations associated to the points in Γ_0 separate $C_0(L)$. Thus $\overline{\Gamma}_0 = L$ and L is dense-separable as Γ was arbitrary. Note that $x \in \overline{\Gamma_0 \cap \mathrm{int}(D(L))}$, so that $\mathrm{int}(D(L))$ is countably perfect. $\square$

We have not been able to construct a (dense-separable) compact K such that $C(K)$ has CSP but fails property (C). Observe that the Čech-Stone compactification $\beta\omega$ is dense-separable as $n \in \omega$ are isolated in $\beta\omega$. Since $C(\beta\omega) = \ell^\infty$, we conclude that dense-separability of K is not sufficient for $C(K)$ to have CSP.

The following result produces plenty of examples of non-separable CSP spaces.

Theorem 3.6. *Let K be a scattered and countably tight compact. Then $C(K)$ has CSP if and only if K is separable.*

Proof. The only if part follows from Proposition 3.5. Towards the other implication, recall that a scattered compact K is countably tight if and only if $C(K)$ has property (C) (see [14, Cor.4.1]). If K is separable, then a standard argument using the point evaluations gives that $C(K)^*$ is ω^*-separable, so that Proposition 3.1 can be applied. $\square$

For example the one-point compactified rational sequence topology (see [17, p.87]) is scattered, countably tight, separable and non-metrizable.

Analogous to the open question about preservation of CSP in finite sums is the open question about the preservation of dense-separability in finite products of topological spaces. The following fact, however, is easy to verify.

Remark 3.7. Let A and B be topological spaces. If A is dense-separable and B has a countable π-base, then $A \times B$ is dense-separable.

3.2. **Auxiliary results: Intersections of distended subspaces.** In addition to property (C) we will treat another type of condition concerning intersections of convex sets, which seems to be closely related to (C). Throughout this subsection let X be a Banach space and κ an uncountable regular ordinal. We denote here by $\{Z_\sigma\}_{\sigma<\kappa}$ a family of closed subspaces of X such that $Z_\alpha \supsetneq Z_\beta$ for $\alpha < \beta < \kappa$ and $\bigcap_{\sigma<\kappa} Z_\sigma = \{0\}$ and we impose the existence of such family for X. This actually excludes X from CSP class, as it will turn out in the next section.

Some subsequent results here depend on the following question:
Given X and $\{Z_\sigma\}_{\sigma<\kappa}$ as above, is $\bigcap_{\sigma<\kappa}(\mathbf{B}_\mathrm{X} + Z_\sigma)$ bounded?
At first glance the answer may appear to be positive for all Banach spaces X. For example, it is easy to see that if X is reflexive, then $\bigcap_{\sigma<\kappa}(\mathbf{B}_\mathrm{X}+Z_\sigma) = \mathbf{B}_\mathrm{X}$. However, next we present an example of a space for which the answer to the above question is negative. Define a function $|||\cdot|||\colon \ell^\infty(\omega_1) \to [0,\infty]$ by

$$|||(x_\alpha)_{\alpha<\omega_1}||| = ||(x_\alpha)_{\alpha<\omega_1}||_{\ell^\infty(\omega_1)} + \sum_{n<\omega} n \limsup_{i\to\omega_1} |x_{\omega i+n}| \quad \text{(ordinal arithmetic)}.$$

Then $(\mathrm{X}, |||\cdot|||)$, where $\mathrm{X} = \{x \in \ell^\infty(\omega_1) : \ |||x||| < \infty\}$, is clearly a Banach space. Let $E_\sigma = \{(x_\alpha)_{\alpha<\omega_1} \in \mathrm{X} : \ x_\alpha = 0 \text{ for } \alpha < \sigma\}$ for $\sigma < \omega_1$. We denote by $\mathbf{1}_A\colon [0,\omega_1) \to \{0,1\}$ the characteristic function of a given subset $A \subset \omega_1$. Note

that $\mathbf{1}_{[0,\sigma]} \in \mathbf{S}_{(\mathrm{X},|||\cdot|||)}$ for all $\sigma < \omega_1$. Hence $\mathbf{1}_{\{\omega i+n | i<\omega_1\}} \in \bigcap_{\sigma<\omega_1}(\mathbf{B}_{(\mathrm{X},|||\cdot|||)} + E_\sigma)$ for all $n < \omega$. Consequently $\bigcap_{\sigma<\omega_1}(\mathbf{B}_{(\mathrm{X},|||\cdot|||)} + E_\sigma)$ is unbounded.

For convenience we denote by $(\mathcal{B})$ the class of Banach spaces X satisfying that $\bigcap_{\sigma<\kappa} \mathbf{B}_\mathrm{X} + Z_\sigma$ is bounded for any $\{Z_\sigma\}_{\sigma<\kappa}$ such as above (or trivially if such $\{Z_\sigma\}_{\sigma<\kappa}$ does not exist at all).

Proposition 3.8. *For any* X *we have that* $\bigcap_{\epsilon>0}\bigcap_{\sigma<\kappa}(\epsilon\mathbf{B}_\mathrm{X} + Z_\sigma) = \{0\}$.

Proof. Observe that $\bigcap_{\sigma<\kappa}(\epsilon\mathbf{B}_\mathrm{X} + Z_\sigma)$ is a symmetric convex set, which contains 0.

First we wish to check that $\bigcap_{\epsilon>0}\bigcap_{\sigma<\kappa}(\epsilon\mathbf{B}_\mathrm{X} + Z_\sigma)$ does not contain any non-trivial linear subspace L. Indeed, if $\bigcap_{\sigma<\kappa}(\mathbf{B}_\mathrm{X} + Z_\sigma)$ contains $L = [x]$ for some $x \in \mathbf{S}_\mathrm{X}$, then $\mathrm{dist}(kx, Z_\sigma) \leq 1$ for all $k < \omega$ and $\sigma < \kappa$. Observe that

$$\epsilon\mathbf{B}_\mathrm{X} + Z_\sigma = \epsilon\mathbf{B}_\mathrm{X} + \epsilon Z_\sigma = \epsilon(\mathbf{B}_\mathrm{X} + Z_\sigma) \tag{3.1}$$

for all $\epsilon > 0$ and $\sigma < \kappa$. Thus, by putting $\epsilon = \frac{1}{k}$ we obtain that $\mathrm{dist}(x, Z_\sigma) = \frac{\mathrm{dist}(kx,Z_\sigma)}{k} < \frac{1}{k}$ for all $k < \omega$ and $\sigma < \kappa$. Since Z_σ are closed subspaces, we get that $x \in Z_\sigma$ for all $\sigma < \kappa$. This contradicts the fact that $\bigcap_{\sigma<\kappa} Z_\sigma = \{0\} \not\ni x$ and hence X does not contain any non-trivial linear subspace L.

Now, let L be a 1-dimensional subspace and write $l = L \cap \bigcap_{\sigma<\kappa}(\mathbf{B}_\mathrm{X} + Z_\sigma)$. Since $\bigcap_{\sigma<\kappa}(\mathbf{B}_\mathrm{X} + Z_\sigma)$ is convex, symmetric and does not contain L, we obtain that l is bounded. Hence $L \cap \bigcap_{\epsilon>0}\epsilon\bigcap_{\sigma<\kappa}(\mathbf{B}_\mathrm{X} + Z_\sigma) = \{0\}$. Since L was arbitrary, we obtain that $\bigcap_{\epsilon>0}\bigcap_{\sigma<\kappa}(\epsilon\mathbf{B}_\mathrm{X} + Z_\sigma) = \{0\}$. $\square$

Note that in the above proof we did not need that κ is uncountable or regular. Next we will give results towards applications of subsequent Lemma 3.12, which is our main technical machinery.

Proposition 3.9. *Suppose that the dual of* X *satisfies the condition* (4) *of Proposition 3.1. Then* X *is a member of* $(\mathcal{B})$.

Proof. Let $\{Z_\sigma\}_{\sigma<\kappa}$ be a strictly nested sequence of closed subspaces of X such that $\bigcap_{\sigma<\kappa} Z_\sigma = \{0\}$. Observe that $\overline{\mathrm{span}}^{\omega^*}(\bigcup_{\sigma<\kappa} Z_\sigma^\perp) = \overline{\bigcup_{\sigma<\kappa} Z_\sigma^\perp}^{\omega^*} = \mathrm{X}^*$. Fix $f \in \mathrm{X}^*$. Since κ is regular, condition (4) of Proposition 3.1 yields that there is $\sigma_0 < \kappa$ such that $f \in Z_\sigma^\perp$ whenever $\sigma_0 \leq \sigma < \kappa$. Thus $f(\bigcap_{\sigma<\kappa}\mathbf{B}_\mathrm{X} + Z_\sigma) \subset [-||f||, ||f||]$. Note that $\bigcap_{\sigma<\kappa}\mathbf{B}_\mathrm{X} + Z_\sigma$ is weakly bounded as f was arbitrary. Hence $\bigcap_{\sigma<\kappa}\mathbf{B}_\mathrm{X} + Z_\sigma$ is bounded according to the uniform boundedness principle. $\square$

Proposition 3.10. *Suppose that the dual of* X *satisfies the condition* (4) *of Proposition 3.1, that* $\{Y_\sigma\}_{\sigma<\kappa}$ *is a nested family of affine subspaces, where* κ *is an uncountable regular cardinal, and that* $\bigcap_{\sigma<\kappa} Y_\sigma = \{y\}$. *If* $y_\sigma \in Y_\sigma$ *for* $\sigma < \kappa$, *then* $y_\sigma \to y$ *weakly as* $\sigma \to \kappa$.

Proof. Suppose that $u + Z_1 = Y_1 \supset Y_2 = u + v + Z_2$ are closed affine subspaces, where $u, v \in \mathrm{X}$ and $Z_1, Z_2 \subset \mathrm{X}$ are closed subspaces. Then $v \in Z_{\sigma_1}$ and $Z_{\sigma_2} \subset Z_{\sigma_1}$. Indeed, if $Z_1 \not\supset Z_2$, then there is no $u + v \in \mathrm{X}$ such that $u + Z_1 \supset u + v + Z_2$. On the other hand, if $v \notin Z_1$, then $v + Z_2 \not\subset Z_1$, so that $u + v + Z_2 \not\subset u + \mathrm{Z}_1$.

Let $\{Z_\sigma\}_{\sigma<\kappa}$ be the nested family of closed subspaces of X corresponding to $\{Y_\sigma\}_{\sigma<\kappa}$. It follows that $y_\sigma \in y + Z_\sigma$ for $\sigma < \kappa$ and $\bigcap_{\sigma<\kappa} Z_\sigma = \{0\}$.

Fix $f \in \mathrm{X}^*$. Observe that $\overline{\mathrm{span}}^{\omega^*}(\bigcup_{\sigma<\kappa} Z_\sigma^\perp) = \mathrm{X}^*$ and $Z_\sigma^\perp$ is ω^*-closed for $\sigma < \kappa$. By using condition (4) of Proposition 3.1 and the fact that $\mathrm{cf}(\kappa) > \omega$, we obtain that there is $\sigma_0 < \kappa$ such that $f \in Z_{\sigma_0}^\perp$. Then $f(y_\sigma - y) = 0$ for $\sigma \geq \sigma_0$, so that we have the claim as f was arbitrary. $\square$

Theorem 3.11. *Suppose that* X *satisfies the following condition: For each nested family of closed affine subspaces* $\{Y_\sigma\}_{\sigma<\kappa}$ *it holds that* $\bigcap_{\sigma<\kappa} Y_\sigma \neq \emptyset$. *Then* X *is a member of* $(\mathcal{B})$.

Before giving the proof, we note that $\sup_{\sigma<\kappa} \mathrm{dist}(0, Y_\sigma) < \infty$, since κ is uncountable and regular.

Proof. Let κ and $\{Z_\sigma\}_{\sigma<\kappa}$ be as in the beginning of this subsection. Since $\bigcap_{\sigma<\kappa} Z_\sigma = \{0\}$, we may define a norm $|||\cdot|||\colon \mathrm{X} \to \mathbb{R}$ by $|||x||| = \sup_{\sigma<\kappa} \mathrm{dist}(x, Z_\sigma)$. Observe that $|||x||| \leq ||x||$ for $x \in \mathrm{X}$.

We aim to show that the norms $||\cdot||$ and $|||\cdot|||$ are equivalent. After this has been established, it follows that there is $C > 0$ such that $||x|| \leq C$ whenever $\mathrm{dist}(x, Z_\sigma) \leq 1$ for $\sigma < \kappa$. Actually, it suffices to check that $|||\cdot|||$ is complete. Indeed, in such case the Banach open mapping principle yields that $\mathbf{I}\colon (\mathrm{X}, ||\cdot||) \to (\mathrm{X}, |||\cdot|||)$ is an isomorphism.

Let $(x_n)_{n<\omega} \subset \mathrm{X}$ be a $|||\cdot|||$-Cauchy sequence, where $x_0 = 0$. Denote $\widehat{x}_n^\sigma = x_n + Z_\sigma \in \mathrm{X}/Z_\sigma$ for $n < \omega,\ \sigma < \kappa$. Observe that

$$|||x_i - x_j||| = \sup_{\sigma<\kappa} ||\widehat{x}_i^\sigma - \widehat{x}_j^\sigma||_{\mathrm{X}/Z_\sigma} \quad \text{for } i, j \in \omega.$$

Since $\mathrm{cf}(\kappa) > \omega$, there is for $(i, j) \in \omega \times \omega$ an ordinal $\alpha_{i,j} < \kappa$ such that

$$|||x_i - x_j||| = ||\widehat{x}_i^{\alpha_{i,j}} - \widehat{x}_j^{\alpha_{i,j}}||_{\mathrm{X}/Z_{\alpha_{i,j}}}.$$

Observe that $\alpha \doteq \sup_{i,j\in\omega} \alpha_{i,j} < \kappa$. Next we regard $\sigma \in [\alpha, \kappa)$.

Thus the sequence $(\widehat{x}_n^\sigma)_{n<\omega} \subset \mathrm{X}/Z_\sigma$ is Cauchy for all $\sigma \in [\alpha, \kappa)$. Since X/Z_σ is a Banach space, we obtain that there is $y^\sigma \in \mathrm{X}/Z_\sigma$ such that $\widehat{x}_n^\sigma \to y^\sigma$ as $n \to \infty$ in X/Z_σ. Moreover, by the selection of α we get $\lim_{n\to\infty} |||x_n||| = \lim_{n\to\infty} ||\widehat{0}^\sigma - \widehat{x}_n^\sigma||_{\mathrm{X}/Z_\sigma}$, since $x_0 = 0$.

We may regard $y^\sigma \doteq Y_\sigma \doteq v_\sigma + Z_\sigma \subset \mathrm{X}$ as affine subspaces, where $v_\sigma \in \mathrm{X}$ for $\sigma \in [\alpha, \kappa)$. Similar interpretation for $\widehat{x}_n^\sigma \in \mathrm{X}$ yields the following: Since $\widehat{x}_n^{\sigma_1} = \widehat{x}_n^{\sigma_2}/Z_{\sigma_1}$ for $n < \omega,\ \alpha \leq \sigma_1 \leq \sigma_2 < \kappa$, we obtain that $y^{\sigma_1} = y^{\sigma_2}/Z_{\sigma_1}$ by the continuity of the quotient map $\mathrm{X}/Z_{\sigma_2} \to \mathrm{X}/Z_{\sigma_1}$. Thus $Y_{\sigma_1} = Y_{\sigma_2} + Z_{\sigma_1}$, where $\alpha \leq \sigma_1 \leq \sigma_2 < \kappa$. Note that $\mathrm{dist}(0, Y_\sigma) = \lim_{n\to\infty} |||x_n|||$ for $\sigma \in [\alpha, \kappa)$ by the selection of α. According to the assumptions $\bigcap_{\sigma<\kappa} Y_\sigma \neq \emptyset$ and let us choose $x \in \bigcap_{\sigma<\kappa} Y_\sigma$ (even though it turns out promptly that this set is a singleton). Observe that $y^\sigma = \widehat{x}^\sigma \doteq x + Z_\sigma$ for $\sigma \in [\alpha, \kappa)$. Thus $\sup_{\sigma\in[\alpha,\kappa)} ||\widehat{x}_n^\sigma - \widehat{x}^\sigma||_{\mathrm{X}/Z_\sigma} \to 0$ as $n \to \infty$. Hence $x_n \xrightarrow{|||\cdot|||} x$ as $n \to \infty$, which completes the proof. $\square$

Lemma 3.12. *Let* X *be a Banach space,* $\mathrm{Y} \subset \mathrm{X}$ *a closed subspace and* κ *an uncountable regular cardinal. Let* $Z_\sigma \subset \mathrm{X}$ *be closed subspaces, for* $\sigma < \kappa$, *which satisfy* $Z_\alpha \subsetneq Z_\beta$ *for* $\beta < \alpha < \kappa$ *and* $\bigcap_{\sigma<\kappa} Z_\sigma = \{0\}$. *Then the following facts hold:*

(i) *If* $\mathrm{dens}(\mathrm{Y}) < \kappa$ *and* $\mathrm{X} \in (\mathcal{B})$, *then* $\mathrm{Y} = \bigcap_{\sigma<\kappa} \overline{\mathrm{Y} + Z_\sigma}$.

(ii) *If* $\mathrm{dens}(\mathrm{Y}) < \kappa$, *then there exists* $\theta < \kappa$ *such that* $Z_\theta \cap \mathrm{Y} = \{0\}$.

Proof. Let us treat the claim (i). Let $x \in \bigcap_{\sigma<\kappa} \overline{\mathrm{span}}(\mathrm{Y} \cup Z_\sigma)$ and $\epsilon > 0$. Thus there is a cofinal sequence $\{\sigma_\alpha\}_{\alpha<\kappa} \subset \kappa$ and families $\{y_\alpha\}_{\alpha<\kappa} \subset \mathrm{Y}$ and $\{z_\alpha\}_{\alpha<\kappa} \subset \mathrm{X}$ such that $z_\alpha \in Z_{\sigma_\alpha}$ and $||x - (y_\alpha + z_\alpha)|| < \frac{\epsilon}{2}$ for each $\alpha < \kappa$.

Note that we have $\bigcap_{\beta<\kappa} \overline{\{y_\alpha | \beta < \alpha < \kappa\}} \neq \emptyset$, because $\overline{\{y_\alpha | \beta < \alpha < \kappa\}}_{\beta<\kappa}$ is a decreasing sequence of closed sets in Y, where κ is regular, and the Lindelöf number of Y is less than κ.

Let $y^{(\epsilon)} \in \bigcap_{\beta<\kappa} \overline{\{y_\alpha | \beta < \alpha < \kappa\}}$ depending on the choice of ϵ. Hence we may pick a cofinal sequence $\{\alpha_\delta\}_{\delta<\kappa} \subset \kappa$ such that $||y^{(\epsilon)} - y_{\alpha_\delta}|| < \frac{\epsilon}{2}$ for each $\delta < \kappa$. This means that

$$||x - (y^{(\epsilon)} + z_{\alpha_\delta})|| \leq ||y^{(\epsilon)} - y_{\alpha_\delta}|| + ||x - (y_{\alpha_\delta} + z_{\alpha_\delta})|| < \epsilon$$

for $\delta < \kappa$. We get that $x - y^{(\epsilon)} \in z_{\alpha_\delta} + \epsilon \mathbf{B}_{\mathrm{X}}$ and in particular

$$x - y^{(\epsilon)} \in Z_{\sigma_{\alpha_\delta}} + \epsilon \mathbf{B}_{\mathrm{X}} \text{ for } \delta < \kappa. \tag{3.2}$$

Since $\{Z_\sigma\}_{\sigma<\kappa}$ decreases to $\{0\}$ and $\sup_{\delta<\kappa} \alpha_\delta = \kappa$, we obtain that $\bigcap_{\delta<\kappa} Z_{\sigma_{\alpha_\delta}} = \{0\}$.

Since X belongs to $(\mathcal{B})$, it follows by (3.1) that

$$\lim_{\epsilon \to 0^+} \mathrm{diam}(\bigcap_{\delta<\kappa} (\epsilon \mathbf{B}_{\mathrm{X}} + Z_{\sigma_{\alpha_\delta}})) = 0.$$

Since $\epsilon > 0$ was arbitrary in (3.2), we conclude that $\mathrm{dist}(x, \mathrm{Y}) = 0$, that is $x \in \mathrm{Y}$ as Y is closed. This completes the proof of claim (i).

Let us check claim (ii). Since $\mathrm{dens}(\mathrm{Y}) < \kappa$, the Lindelöf number of $\mathrm{Y} \setminus \{0\}$ is less than κ. It follows by the regularity of κ, that there cannot exist a decreasing sequence $\{(Z_\alpha \cap \mathrm{Y}) \setminus \{0\}\}_{\alpha<\kappa}$ of non-empty closed sets in $\mathrm{Y} \setminus \{0\}$. □

We note that in the above lemma the additional assumptions in (i) cannot be removed. Indeed, consider closed subspaces

$$E_\alpha = \{(x_i) \in \ell^\infty(\omega_1) | x_i = 0, \text{ for } i \leq \alpha\}, \quad \alpha < \omega_1$$

of $\ell^\infty(\omega_1)$. Then $\bigcap_{\alpha<\omega_1} \ell_c^\infty(\omega_1) + E_\alpha = \ell^\infty(\omega_1)$. Note that $\mathrm{dens}(\ell_c^\infty(\omega_1)) = 2^\omega$ and $\mathrm{dens}(\ell^\infty(\omega_1)) = 2^{\omega_1}$.

4. Combinatorial approach to CSP spaces

Recall that if X has CSP, then ω^*−$\mathrm{dens}(\mathrm{X}^*) = \omega$. This in turn implies $\mathrm{dens}(\mathrm{X}) \leq |\mathrm{X}| \leq 2^\omega$. Recall that $2^\omega < \aleph_\omega$ is consistent with ZFC. For this reason some of the results here, which involve assumption about the density of the space, can actually be thought of as consistency results.

For a given Banach space X we define the *cofinality range* of X, cr(X) for short, as the set of all infinite regular cardinals κ satisfying that there exists a family $\{E_\sigma\}_{\sigma<\kappa}$ of closed subspaces of X such that $\bigcap_{\sigma<\kappa} E_\sigma = \{0\}$ and $E_\alpha \subsetneq E_\beta$ whenever $\alpha < \beta < \kappa$. Observe that each $\kappa \in \mathrm{cr}(\mathrm{X})$ satisfies $\kappa \leq \mathrm{dens}(\mathrm{X})$.

Theorem 4.1. *Let* X *be a Banach space such that* $\mathrm{dens}(\mathrm{X}) < \aleph_\omega$. *Then* X *has CSP if and only if* $\mathrm{cr}(\mathrm{X}) = \{\omega\}$.

Proof. Let us first check the easier 'only if' part. Suppose that for $\{E_\sigma\}_{\sigma<\kappa}$, as above, there does not exist $(\sigma_n)_{n<\omega}$ such that $\bigcap_{n<\omega} E_{\sigma_n} = \{0\}$, or equivalently $\sup_{n<\omega} \sigma_n = \kappa$. Note that $\bigcup_{\sigma<\kappa} E_\sigma^\perp \subset \mathrm{X}^*$ separates X by the Hahn-Banach theorem. Clearly this set has no countable separating subset, so that X fails CSP.

To check the 'if' part assume that X satisfies $\mathrm{dens}(\mathrm{X}) < \aleph_\omega$ and X fails CSP. We aim to show that in such case $\mathrm{cr}(\mathrm{X}) \neq \{\omega\}$. According to (2.1) there is an uncountable separating family $\mathcal{F} \subset \mathrm{X}^*$, whose all separating subfamilies have the same cardinality, say $\kappa \geq \omega_1$. Write $\mathcal{F} = \{f_\alpha\}_{\alpha<\kappa}$. Put $F_\sigma = \bigcap_{\alpha<\sigma} \mathrm{Ker} f_\alpha$ for all $\sigma < \kappa$. Clearly this gives a (not necessarily strictly) nested family of closed subspaces. Note that $\bigcap_{\sigma<\kappa} F_\sigma = \{0\}$.

Let $\phi(0) = 0$ and we define recursively

$$\phi(\alpha) = \min\{\beta < \kappa : \bigcap_{\gamma<\alpha} \operatorname{Ker}(f_{\phi(\gamma)}) \not\subset \operatorname{Ker}(f_\beta)\}.$$

Thus, putting $E_0 = \mathrm{X}$, $E_\sigma = \bigcap_{\alpha<\sigma} \operatorname{Ker}(f_{\phi(\alpha)})$ for $0 < \sigma < \kappa$ gives a strictly nested family and $\bigcap_{\sigma<\lambda} E_\sigma = \{0\}$ for some ordinal $\lambda \le \kappa$. Note that by the construction of ϕ it holds that $\{f_{\phi(\alpha)}\}_{\alpha<\lambda}$ is a separating family.

Since $\{f_\alpha\}_{\alpha<\kappa}$ does not have a separating subfamily of cardinality less than κ, we conclude that $\lambda = \kappa$ above. As $\operatorname{dens}(\mathrm{X}) < \aleph_\omega$ we obtain that $\lambda = \kappa < \aleph_\omega$ is a regular cardinal and $\{E_\alpha\}_{\alpha<\lambda}$ is the required family witnessing that $\omega_1 \le \lambda = \kappa \in \operatorname{cr}(\mathrm{X})$. □

The above proof yields immediately that CSP $\implies$ $(\mathcal{B})$. Note that there are CSP spaces with an uncountable biorthogonal system. For example, it suffices to consider a non-separable dual space with CSP, such as JT^*, since it is known that non-separable dual spaces have uncountable biorthogonal systems (see [18, Cor.4]).

One could ask if the previous result remains valid if one replaces 'cofinality' by 'cardinality' in the definition of $\operatorname{cr}(\mathrm{X})$. This is not the case as the following example shows; the assumption about the regularity of κ in the definition of $\operatorname{cr}(\mathrm{X})$ is indeed essential:

Example 4.2. If X is a CSP space with a biorthogonal system $\{(x_\alpha, x_\alpha^*)\}_{\alpha<\omega_1}$, then by using the proof of the previous theorem we see that

(1) $F \doteq \bigcap_{\beta<\omega_1} \overline{\operatorname{span}}(\{x_\alpha | \beta < \alpha < \omega_1\}) \neq \{0\}$
(2) X/F is not a CSP space.
(3) There is a strictly nested family $\{E_\sigma\}_{\sigma<\kappa}$ of closed subspaces of X such that $\bigcap_{\sigma<\kappa} E_\sigma = \{0\}$, where $\kappa > \omega_1$.

The family of subspaces in the last condition can be obtained as follows: Let $\{F_\theta\}_{\theta<\lambda}$ be a strictly nested family of closed subspaces of F, where λ is an ordinal and $\bigcap_{\theta<\lambda} F_\theta = \{0\}$. It suffices to put $E_\sigma = \overline{\operatorname{span}}(\{x_\alpha | \sigma \le \alpha < \omega_1\})$ for $\sigma < \omega_1$, $E_{\omega_1} = F$, and $E_{\omega_1+\theta} = F_{\theta+1}$ for $\theta < \lambda$.

We attempt to bring CSP closer to a three space property in the next result.

Theorem 4.3. *Let* X *be a Banach space satisfying* $(\mathcal{B})$ *and* $\operatorname{dens}(\mathrm{X}) < \aleph_\omega$. *Let* $\mathrm{Y} \subset \mathrm{X}$ *be a closed subspace. Then* $\operatorname{cr}(\mathrm{X}) \subset \operatorname{cr}(\mathrm{X}/\mathrm{Y}) \cup \operatorname{dens}(\mathrm{Y})^+$. *In particular, such* X *has CSP if* Y *is separable and* X/Y *has CSP.*

Proof. Observe that $\operatorname{dens}(\mathrm{Y}), \operatorname{dens}(\mathrm{X}/\mathrm{Y}) \in \aleph_\omega$ are regular cardinals. Denote the quotient map $q\colon \mathrm{X} \to \mathrm{X}/\mathrm{Y}$; $q\colon x \mapsto x + \mathrm{Y}$.

Consider a regular cardinal $\kappa > \operatorname{dens}(\mathrm{Y})$ such that $\kappa \in \operatorname{cr}(\mathrm{X})$. Let $\{E_\sigma\}_{\sigma<\kappa}$ be the corresponding strictly nested family of closed subspaces of X with trivial intersection. By Lemma 3.12 (i) we obtain that $\bigcap_{\sigma<\kappa} \overline{\operatorname{span}}(E_\sigma \cup \mathrm{Y}) = \mathrm{Y}$. Thus $\bigcap_{\sigma<\kappa} (\overline{\operatorname{span}}(E_\sigma \cup \mathrm{Y}) + \mathrm{Y}) = \mathrm{Y}$. Hence Fact 1.1 yields that

$$\bigcap_{\sigma<\kappa} \overline{q(E_\sigma)} = \bigcap_{\sigma<\kappa} q(\overline{\operatorname{span}}(E_\sigma \cup \mathrm{Y})) = 0 \in \mathrm{X}/\mathrm{Y}.$$

Note that $E_\sigma \not\subset \mathrm{Y}$ for any $\sigma < \kappa$, since $\{E_\sigma\}_{\sigma<\kappa}$ is strictly nested, κ is regular and $\operatorname{dens}(\mathrm{Y}) < \kappa$. Hence $\{0\} \subsetneq \overline{q(E_\sigma)}$ in X/Y for each $\sigma < \kappa$. Thus by passing to a cofinal subsequence $\{\sigma_\alpha\}_{\alpha<\kappa} \subset \kappa$ such that $\{\overline{q(E_{\sigma_\alpha})}\}_{\alpha<\kappa} \subset \mathrm{X}/\mathrm{Y}$ is strictly nested we obtain that $\kappa \in \operatorname{cr}(\mathrm{X}/\mathrm{Y})$.

For the latter claim recall that according to Theorem 4.1 it holds that $\mathrm{cr}(\mathrm{X}) = \{\omega\}$ if and only if X has CSP. □

4.1. **The Kunen-Shelah properties.** In [4] the Kunen-Shelah properties

$$\mathrm{KS}_7 \implies \mathrm{KS}_6 \implies \ldots \implies \mathrm{KS}_0$$

are discussed. To mention the most important Kunen-Shelah properties in this context, a space X is said to have KS_4, KS_2 or KS_1, if X admits *no* uncountable polyhedron, no uncountable biorthogonal system or no uncountable M-basic sequence, respectively. It follows easily from [4, p.114-119] that a Banach space X with KS_4 has CSP. For example, Kunen and Shelah have provided samples of non-separable KS_4 spaces by assuming CH or $\diamondsuit(\aleph_1)$, see [13, p.1086-1099] and [16]. Since CSP passes to subspaces, we obtain that CSP $\implies$ KS_1 by Proposition 2.7.

It follows from recent results of Todorčević (see [19]) that it is consistent with ZFC that the properties $\mathrm{KS}_7 - \mathrm{KS}_2$ are in fact equivalent to separability. In particular, $\mathrm{KS}_2 \implies$ CSP is consistent.

Theorem 4.4. *Let* X *be a Banach space such that each quotient* X/Y *satisfies* $(\mathcal{B})$ *and* $\mathrm{dens}(\mathrm{X}) < \aleph_\omega$*. If* X *has* KS_2*, then it has CSP.*

We will first prove the following result, which will be applied.

Proposition 4.5. *Let* X *satisfy* $(\mathcal{B})$ *and* KS_2*. Suppose that* $\{x_\alpha\}_{\alpha<\omega_1} \subset \mathrm{X} \setminus \{0\}$ *and* $\{\Gamma_\sigma\}_{\sigma<\omega_1}$ *is a family of cofinal subsets of* ω_1 *such that* $\Gamma_\alpha \supset \Gamma_\beta$ *for* $\alpha < \beta < \omega_1$*. Then*

$$\bigcap_{\sigma<\omega_1} \overline{\mathrm{span}}(\{x_\alpha |\ \alpha \in \Gamma_\sigma\}) \neq \{0\}.$$

Proof. Assume to the contrary that above

$$\bigcap_{\sigma<\omega_1} \overline{\mathrm{span}}(\{x_\alpha |\ \alpha \in \Gamma_\sigma\}) = \{0\}. \tag{4.1}$$

Then according to Lemma 3.12 (ii) for each $\beta < \omega_1$ there is $\sigma < \omega_1$ such that

$$\overline{\mathrm{span}}(\{x_\alpha | \alpha < \beta\}) \cap \overline{\mathrm{span}}(\{x_\alpha | \alpha \in \Gamma_\sigma\}) = \{0\}. \tag{4.2}$$

Let $\sigma(\beta)$ be the least ordinal satisfying (4.2) for $\sigma = \sigma(\beta)$.

Next we will define recursively an uncountable subfamily of $\{x_\alpha\}_{\alpha<\omega_1}$ by using the above notations. Let $\alpha_0 = 0$ and for each $\theta < \omega_1$ let

$$\alpha_\theta = \min\{\gamma \in \Gamma_{\sigma(\sup_{\epsilon<\theta} \alpha_\epsilon)} :\ \gamma > \sup_{\epsilon<\theta} \alpha_\epsilon\}.$$

Note that $\{\alpha_\theta\}_{\theta<\omega_1}$ is an increasing sequence by its construction. Observe that the corresponding family $\{y_\theta\}_{\theta<\omega_1} = \{x_{\alpha_\theta}\}_{\theta<\omega_1}$ satisfies

$$\overline{\mathrm{span}}(\{y_\theta | \theta < \gamma\}) \cap \overline{\mathrm{span}}(\{y_\theta | \gamma \leq \theta\}) = \{0\}$$

for each $\gamma < \omega_1$.

The assumption (4.1) yields that $\bigcap_{\gamma<\omega_1} \overline{\mathrm{span}}(\{y_\theta | \gamma < \theta < \omega_1\}) = \{0\}$. Thus, an application of Lemma 3.12 (i) for $\mathrm{Y} = \overline{\mathrm{span}}(\{y_\theta : \theta < \gamma\})$ and $Z_\gamma = \overline{\mathrm{span}}(\{y_\theta | \gamma < \theta < \omega_1\})$ for countable γ yields the following fact: For all $\gamma < \delta < \omega_1$ one can find the least ordinal $\eta(\gamma, \delta) \in (\delta, \omega_1)$ such that

$$y_\delta \notin \overline{\mathrm{span}}(\{y_\theta |\ \theta \in [0, \gamma] \cup [\eta(\gamma, \delta), \omega_1)\}).$$

Put $\zeta_0 = 0$, $\zeta_1 = 1$ and for each $\alpha \in (1, \omega_1)$ we define recursively

$$\zeta_\alpha = \sup_{\beta<\alpha} \eta(\sup_{\gamma<\beta} \zeta_\gamma, \zeta_\beta) + 1.$$

This defines an increasing sequence $\{\zeta_\alpha\}_{\alpha<\omega_1} \subset \omega_1$ such that $y_{\zeta_\gamma} \notin \overline{\mathrm{span}}(\{y_{\zeta_\theta} | \gamma \neq \theta < \omega_1\})$ for $\gamma < \omega_1$.

In particular $\{y_{\zeta_\alpha}\}_{\alpha<\omega_1}$ is a minimal system. One can select by an application of the Hahn-Banach theorem suitable functionals $g_\alpha \in \mathrm{X}^*$ to obtain a biorthogonal system $\{(y_{\zeta_\alpha}, g_\alpha)\}_{\alpha<\omega_1}$. This contradicts KS_2, so that we have obtained the claim. □

Proof of Theorem 4.4. Suppose that X fails CSP and that $\mathcal{F} \subset \mathbf{S}_{X^*}$ is a separating set without any countable separating subset. Then by using the proof of Theorem 4.1 we obtain the following: There exists a family $\{E_\sigma\}_{\sigma<\lambda}$ of closed subspaces such that $E_\beta \subsetneq E_\alpha$ whenever $\alpha < \beta < \lambda$. Here λ is uncountable but unlike in the proof of Theorem 4.1, here we do not need any control over the intersection $\bigcap_\sigma E_\sigma$ or the cofinality of λ. Moreover, similarly as in the proof of Theorem 4.1, we may choose $\{E_\sigma\}_\sigma$ in the following manner. Namely, that there exists a family $\{f_{\phi(\gamma)}\}_{\gamma<\omega_1} \subset \mathcal{F}$ such that $E_\sigma = \bigcap_{\gamma<\sigma} \mathrm{Ker} f_{\phi(\gamma)}$ for each $\sigma \leq \omega_1$. We may assign for each $\sigma < \omega_1$ such $x_\sigma \in E_\sigma \setminus E_{\sigma+1}$ that $E_{\sigma+1} + [x_\sigma] = E_\sigma$ and $f_\sigma(x_\sigma) = 1$.

By the selection of $\{E_\sigma\}_\sigma$ and canonical identifications we get that

$$E_{\omega_1} = \bigcap_{\gamma<\omega_1} \mathrm{Ker} f_{\phi(\sigma)} \text{ and } f_{\phi(\gamma)} \in (\mathrm{X}/E_{\omega_1})^* = E_{\omega_1}^{\perp}, \quad \text{for } \gamma < \omega_1.$$

Put

$$F = \bigcap_{\beta<\omega_1} \overline{\mathrm{span}}(\{x_\sigma | \beta < \sigma < \omega_1\})$$

and observe that $F \subset E_{\omega_1}$, since $\overline{\mathrm{span}}(\{x_\sigma | \beta < \sigma < \omega_1\}) \subset \bigcap_{\sigma<\beta} \mathrm{Ker}(f_{\phi(\sigma)})$ for $\beta < \omega_1$. Write $\hat{x}_\sigma = x_\sigma + F \in \mathrm{X}/F$ for $\sigma < \omega_1$. Note that

$$\bigcap_{\beta<\omega_1} \overline{\mathrm{span}}(\{\hat{x}_\sigma | \beta < \sigma < \omega_1\}) = \{0\} \subset \mathrm{X}/F. \tag{4.3}$$

Since X/F satisfies $(\mathcal{B})$ by the assumptions, we may apply the proof of Proposition 4.5 with $\Gamma_\sigma = [\sigma, \omega_1]$ for $\sigma < \omega_1$ to obtain that X/F admits a biorthogonal system of length ω_1. This can be lifted to obtain a corresponding biortohogonal system in X. □

Regarding the assumption $\mathrm{dens}(\mathrm{X}) < \aleph_\omega$, note that if $\mathrm{dens}(\mathrm{X}) > 2^\omega$, then $\omega^*-\mathrm{dens}(\mathrm{X}^*) > \omega$ and X fails KS_2 (see e.g. [4, p.97-98]).

5. Conclusions: examples, remarks and renormings

Let us briefly recall the list of CSP spaces mentioned here: Separable spaces, the Johnson-Lindenstrauss spaces JL_0 and JL_2, the duals JT^*, JF^* due to Lindenstrauss-Stegal, Shelah's space S under $\diamondsuit(\aleph_1)$ and $C(K)$ spaces, where K is the double-arrow space, Kunen's compact under CH, or any scattered separable countably tight compact.

Observe that the non-separable spaces above do not admit a system (2.2).

Proposition 5.1. *Let* X *be a Banach space with CSP. Then* X *admits an equivalent uniformly Gateaux (UG) norm if and only if* X *is separable.*

Proof. Each separable space X admits an equivalent UG norm (see [21, Cor.6.3]). Conversely, each space X with an equivalent UG norm is weakly countably determined and thus WLD (see [21, Thm.6.5,Thm.3.8]). As WLD spaces are Plichko ([8, Thm.1]) we obtain by Example 2.8 that X is in fact separable. □

A similar result does not hold for LUR renormings even for dual spaces. The space JT^* is a non-separable CSP space, which admits an equivalent LUR norm by the three space property of LUR renormings (see [21, p.1758,1785]). On the other hand, for Kunen's compact K the space $C(K)$ does not admit a Kadets-Klee and in particular not a LUR norm (see [21, p.1794]).

Often the dual spaces behave better than their underlying spaces, so it is reasonable to restate the following known question.

Problem 5.2. *Does any dual space* X^* *with CSP, i.e. a space* X^* *such that* X *is separable and does not contain* ℓ^1 *isomorphically, admit an equivalent (not necessarily dual) LUR norm?*

A positive answer to the previous question would partly generalize the following result. If X is an Asplund space then X^* has an equivalent (not necessarily dual) LUR norm (see [21, Thm. 7.13]). Note that any non-separable CSP dual space is *not* LUR by ω^*-Kadets-Klee property of X^*.

If X^* has the RNP and CSP, then X is a separable Asplund space by Theorem 3.4 and the duality of the Radon-Nikodym and Asplund properties. This means that X^* must be separable.

Problem 5.3. *Are all spaces* X *with CSP and the RNP in fact separable?*

We would like to emphasize the significance of the following problems.

Problem 5.4. *If* X *and* Y *have CSP, does it follow that* $\mathrm{X} \oplus \mathrm{Y}$ *has CSP? If so, is CSP a three-space property, i.e. does* X *have CSP whenever* X/Y *and* $\mathrm{Y} \subset \mathrm{X}$ *have CSP?*

Problem 5.5. *Does there exist a CSP space without property* (C)*?*

Proposition 5.6. *If* X^* *and* Y^* *are dual spaces with CSP, then* $\mathrm{X}^* \oplus \mathrm{Y}^*$ *has CSP.*

Proof. First we note that the property of being a predual of a CSP space is a three-space property. Indeed, the property of simultaneous separability and non-containment of ℓ^1 isomorphically is a three-space property (see [2, 2.4.h,3.2.d]). Thus we may apply Theorem 3.4.

Now, let X and Y be some preduals of X^* and Y^*, respectively. Since being a predual of a CSP is a three space property, we get that $\mathrm{X} \oplus \mathrm{Y}$ is a predual of a CSP space. The dual $(\mathrm{X} \oplus \mathrm{Y})^*$ is thus a CSP space, which is isomorphically $\mathrm{X}^* \oplus \mathrm{Y}^*$. □

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